table of contents

1 procedure for determining the inverse matrix

The inverse matrix of a matrix A can be determined only if the determinant of the matrix A is different from zero.

The following procedures are used for the calculation of the inverse matrix.

1.1 Calculation of the inverse matrix using the Gaussian algorithm

To the matrix A= $\begin{pmatrix} -3 & 3 & -2 \\ 7 & 5 & 1 \\ 4 & -2 & 2 \end{pmatrix}$
the inverse matrix shall be calculated.

The following scheme can be used for the calculation.

additional Translation: die inverse matrix zu ... lautet = the inverse matrix to ... is ...

Abbildung 1: Exercise 1b - Scheme for calculation of the inverse matrix

-3	3	-2	1	0	0	
7	5	1	0	1	0	
4	-2	2	0	0	1	
1	1	0	1	0	1	1
7	5	1	0	1	0	
4	-2	2	0	0	1	*
1	1	0	1	0	1	*
0	-2	1	-7	1	-7	-7
4	-2	2	0	0	1	
1	1	0	1	0	1	*
0	-2	1	-7	1	-7	
0	-6	2	-4	0	-3	-4
1	1	0	1	0	1	
0	-2	1	-7	1	-7	*
0	0	-1	17	-3	18	-3
1	1	0	1	0	1	
0	-2	0	10	-2	11	1
0	0	-1	17	-3	18	*
1	0	0	6	-1	13/2	1/2
0	-2	0	10	-2	11	*
0	0	-1	17	-3	18	
1	0	0	6	-1	13/2	
0	1	0	-5	1	-11/2	
0	0	1	-17	3	-18	

Die inverse Matrix zu $\begin{pmatrix} -3 & 3 & -2 \\ 7 & 5 & 1 \\ 4 & -2 & 2 \end{pmatrix}$ lautet $\begin{pmatrix} 6 & -1 & \frac{13}{2} \\ -5 & 1 & -\frac{11}{2} \\ -17 & 3 & -18 \end{pmatrix}$

The procedure for the calculation of the inverse matrix is described in detail in the following.

Description of the procedure for the calculation of the inverse matrix

The unit matrix is written right next to the matrix which shall be inverted. A double matrix is achieved. An example is written below.

The goal is to achieve that the unit matrix stands on the left side of the double matrix, after performing elementary row transformations. On the right side of the unit matrix then stands the inverse matrix.

As transformations are allowed:

(1) Multiplying a row with a real number, which must be different from zero
(2) Addition of a row to another row
(3) It is also allowed to multiply a row with a number different from zero and to add it to another row and leave the row itself unchanged.

Example
The folling matrix shall be inverted

$$A = \begin{pmatrix} -3 & 3 & -2 \\ 7 & 5 & 1 \\ 4 & -2 & 2 \end{pmatrix}$$

First step: the unit matrix is written right next to the matrix A

$$\left(\begin{array}{ccc|ccc} -3 & 3 & -2 & 1 & 0 & 0 \\ 7 & 5 & 1 & 0 & 1 & 0 \\ 4 & -2 & 2 & 0 & 0 & 1 \end{array}\right)$$

Addition of the third row to the first row generates a zero in one of the desired locations.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 7 & 5 & 1 & 0 & 1 & 0 \\ 4 & -2 & 2 & 0 & 0 & 1 \end{array}\right)$$

Multiplying the first row with -7 and addition to the second row creates ane more zero

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -7 & 1 & -7 \\ 4 & -2 & 2 & 0 & 0 & 1 \end{array}\right)$$

Multiply the first row with -4 and add it to the to the third row afterwards

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -7 & 1 & -7 \\ 0 & -6 & 2 & -4 & 0 & -3 \end{array}\right)$$

Multiply the second row with -3 and add the result to the third row

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -7 & 1 & -7 \\ 0 & 0 & -1 & 17 & -3 & 18 \end{array}\right)$$

Adding the third line to the second line results

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & -2 & 0 & 10 & -2 & 11 \\ 0 & 0 & -1 & 17 & -3 & 18 \end{array}\right)$$

Multiply the second row with $\frac{1}{2}$ and add the result to the first row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -1 & \frac{13}{2} \\ 0 & -2 & 0 & 10 & -2 & 11 \\ 0 & 0 & -1 & 17 & -3 & 18 \end{array}\right)$$

Division of the second row by -2 and division of the third row by -1 results in

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -1 & \frac{13}{2} \\ 0 & 1 & 0 & -5 & 1 & -\frac{11}{2} \\ 0 & 0 & 1 & -17 & 3 & -18 \end{array}\right)$$

The procedure ends now because the unit matrix stands at the left side of the double matrix.

The inverse matrix of A is $A^{-1} = \begin{pmatrix} 6 & -1 & \frac{13}{2} \\ -5 & 1 & -\frac{11}{2} \\ -17 & 3 & -18 \end{pmatrix}$

1.2 Calculation of the inverse matrix using the adjoint matrix

1.2.1 The procedure for a (3,3) matrix

To a matrix A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

a matrix K is determined whose elements are defined as follows:

$K_{ij} = (-1)^{i+j} det(A_{ij})$

The matrix A_{ij} is derived from the matrix A by removing the row i and the column j from A.

Example: A_{11} is build by A by removing the first row and the first column.

$A_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$

$det(A_{11})$ notes the determinant of the matrix A_{11}

The matrix element K_{11} is calculated as $(-1)^{1+1} det(A_{11}) = a_{22}a_{33} - a_{23}a_{32}$

End of the example

The matrix $A_{adj} = K^T$ is named the adjoint matrix to A.

K^T is the transposed matrix of K.

The transposed matrix K^T is derived by swapping rows and columns of K.

With the adjoint matrix A_{adj} the inverse matrix A^{-1} to a matrix A can be calculated, if $detA \neq 0$

$$A^{-1} = \frac{1}{detA} A_{adj}$$

1.2.2 Exercise to determine the inverse matrix of a (3,3) matrix using the adjoint matrix

Sei A= $\begin{pmatrix} -3 & 3 & -2 \\ 7 & 5 & 1 \\ 4 & -2 & 2 \end{pmatrix}$

Calculating the matrix K

$K_{11} = (-1)^{1+1} detA_{11} = det \begin{pmatrix} 5 & 1 \\ -2 & 2 \end{pmatrix} = 10 + 2 = 12$

$K_{12} = (-1)^{1+2} det A_{12} = -det \begin{pmatrix} 7 & 1 \\ 4 & 2 \end{pmatrix} = -(14-4) = -10$

$K_{13} = (-1)^{1+3} det A_{13} = det \begin{pmatrix} 7 & 5 \\ 4 & -2 \end{pmatrix} = -14-20 = -34$

$K_{21} = (-1)^{2+1} det A_{21} = -det \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix} = -(6-4) = -2$

$K_{22} = (-1)^{2+2} det A_{22} = det \begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix} = -6+8 = 2$

$K_{23} = (-1)^{2+3} det A_{23} = -det \begin{pmatrix} -3 & 3 \\ 4 & -2 \end{pmatrix} = -(6-12) = 6$

$K_{31} = (-1)^{3+1} det A_{31} = det \begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix} = 3+10 = 13$

$K_{32} = (-1)^{3+2} det A_{31} = -det \begin{pmatrix} -3 & -2 \\ 7 & 1 \end{pmatrix} = -(-3+14) = -11$

$K_{33} = (-1)^{3+3} det A_{31} = det \begin{pmatrix} -3 & 3 \\ 7 & 5 \end{pmatrix} = -15-21 = -36$

It follows:

K= $\begin{pmatrix} 12 & -10 & -34 \\ -2 & 2 & 6 \\ 13 & -11 & -36 \end{pmatrix}$

$K^T = \begin{pmatrix} 12 & -2 & 13 \\ -10 & 2 & -11 \\ -34 & 6 & -36 \end{pmatrix}$

Calculating the determinant of A:

For a matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

the determinant $detA$ is written as

$detA = |A| = a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32}$

$-a_{13} \cdot a_{22} \cdot a_{31} - a_{12} \cdot a_{21} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32}$

For A= $\begin{pmatrix} -3 & 3 & -2 \\ 7 & 5 & 1 \\ 4 & -2 & 2 \end{pmatrix}$

it follows:

$detA = (-3) \cdot 5 \cdot 2 + 3 \cdot 1 \cdot 4 + (-2) \cdot 7 \cdot (-2)$

$2 \cdot 5 \cdot 4 + (-3) \cdot 7 \cdot 2 + 3 \cdot 1 \cdot (-2)$

$= -30 + 12 + 28 + 40 - 42 - 6 = 2$

Result

$$A^{-1} = \frac{1}{detA} A_{adj}$$

$$= \frac{1}{2}\begin{pmatrix} 12 & -2 & 13 \\ -10 & 2 & -11 \\ -34 & 6 & -36 \end{pmatrix} = \begin{pmatrix} 6 & -1 & \frac{13}{2} \\ -5 & 1 & -\frac{11}{2} \\ -17 & 3 & -18 \end{pmatrix}$$

1.2.3 Determination of the adjoint matrix for a (2,2) matrix

For a (2,2)-Matrix the procedure for calculating the adjoint matrix can be simplified.

Let B= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then it is $detB = ad - bc$

The elements of K are calculated as follows:

$K_{11} = (-1)^{1+1} det(B_{11})$

with $B_{11} = d$. If the first row and the first column of B are removed, only the element d remains. The determinant of a matrix, consisting only of one element, is just this element.

It follows $K_{11} = d$

According the foregoing procedure one gets:

$K_{12} = (-1)^{1+2} c = -c$
$K_{21} = (-1)^{2+2} b = -b$
$K_{22} = (-1)^{2+2} a = a$

It follows K= $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

and therefore $K^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

It follows: $B^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

1.2.4 Exercise for calculating the inverse matrix of a (2,2) matrix using the adjoint matrix

For the matrix $A = \begin{pmatrix} 5 & \lambda \\ 2 & 4 \end{pmatrix}$
the adjoint matrix is written as:

$adj(A) = \begin{pmatrix} 4 & -\lambda \\ -2 & 5 \end{pmatrix}$

The determinant of A is calculated as

$$detA = det\begin{pmatrix} 5 & \lambda \\ 2 & 4 \end{pmatrix} = 20 - 2\lambda$$

For the calculation of the inverse matrix it is assumed, $\lambda \neq 10$. Otherwise the determinant von A results in zero and the inverse matrix does not exist.

The inverse matrix to A, A^{-1}, is calculated as

$$A^{-1} = \frac{1}{20 - 2\lambda} \cdot \begin{pmatrix} 4 & -\lambda \\ -2 & 5 \end{pmatrix}$$

Inhaltsverzeichnis