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Matrices, Gaussian algorithm

1 Determination of the inverse matrix

On this page, matrices and vectors are printed bold to distinguish them from numbers.

Considered is the linear system of equations

$$5x_1 + 6x_2 = 7$$
$$3x_1 + 4x_2 = 9$$

1.1 Understanding matrix vector multiplication

Let $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$

$\mathbf{A}$ is the coefficient matrix of the above given linear system of equations, $\mathbf{x}$ is the solution vector.

The linear system of equations can be written as a matrix-vector multiplication as follows

$\mathbf{Ax} = \mathbf{b}$
The operation symbol between the matrix $\mathbf{A}$ and the vector $\mathbf{x}$ is usually not written.

1.1.1 Multiplying a matrix with a vector

The multiplication of the matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is defined as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 \end{pmatrix}$$

This means for the task:

$$\begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \cdot x_1 + 6 \cdot x_2 \\ 3 \cdot x_1 + 4 \cdot x_2 \end{pmatrix}$$

The result of the multipliction is the vector $\mathbf{b}$.

$$\mathbf{b} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

that is, it must apply:

$$\begin{pmatrix} 5 \cdot x_1 + 6 \cdot x_2 \\ 3 \cdot x_1 + 4 \cdot x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

Vectors are equal if their components are equal.

That means for the latest specified equation

$$
\begin{aligned}
5x_1 + 6x_2 &= 7 \\
3x_1 + 4x_2 &= 9
\end{aligned}
$$

This is precisely the original system.

Why this whole considerations?

1.1.2 Determination of the solution of the LGS using the inverse matrix

I want to show how to come to a solution of the equation system using the inverse matrix of the coefficient matrix.

If the matrix $\mathbf{A}$ is invertible, the inverse matrix $\mathbf{A^{-1}}$ exists.

The equation $\mathbf{Ax} = \mathbf{b}$ can be multiplied from left with $\mathbf{A^{-1}}$

$\mathbf{A^{-1}}(\mathbf{Ax}) = \mathbf{A^{-1}b}$

This multiplication is associative, i.e. it is

$\mathbf{A^{-1}}(\mathbf{Ax}) = (\mathbf{A^{-1}A})\mathbf{x}$

The result of the product $(\mathbf{A^{-1}A})$ is the identity matrix. In the case of this task it is a matrix consisting of 2 rows and 2 columns:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The multiplication of the identity matrix with the vector $\mathbf{x}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

yields the vector $\mathbf{x}$.

The equation
$\mathbf{A^{-1}}(\mathbf{Ax}) = \mathbf{A^{-1}b}$

reduces to

$\mathbf{x} = \mathbf{A^{-1}b}$

This equation shall be used to solve the task.
The inverse matrix $\mathbf{A^{-1}}$ must be determined first.

For this purpose, there exists a solution process that is presented in the following.

1.2 Calculation of the inverse matrix

To the matrix $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$

a matrix $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

has to be calculated, so that the following applies:

$$\begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The elements of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ have the following form:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

1.2.1 Matrix-Matrix-Multiplication

The Matrix-Matrix Multiplication between $\mathbf{A}$ and $\mathbf{B}$, giving $\mathbf{C}$, is defined as follows:

The element c_{ik} of the result $\mathbf{C}$ is given, by multiplying the row i of $\mathbf{A}$ with the column k of $\mathbf{B}$

Row i of $\mathbf{A}$, $\begin{pmatrix} a_{i1} & a_{i2} \end{pmatrix}$ can be considered as a row vector, column k of $\mathbf{B}$,
$\begin{pmatrix} b_{1k} \\ b_{2k} \end{pmatrix}$ as a column vector.
The multiplication of these two vectors, considered as a scalar product, gives $a_{i1}b_{1k} + a_{i2}b_{2k}$

It follows:

$c_{11} = a_{11}b_{11} + a_{12}b_{21}$
$c_{12} = a_{11}b_{12} + a_{12}b_{22}$
$c_{21} = a_{21}b_{11} + a_{22}b_{21}$
$c_{22} = a_{21}b_{12} + a_{22}b_{22}$

for the multiplication

$$\begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

this means

$5b_{11} + 6b_{21} = 1$, first row of $\mathbf{A}$ multiplied with the first column of $\mathbf{B}$ yields $c_{11} = 1$
$5b_{12} + 6b_{22} = 0$, first row of $\mathbf{A}$ multiplied with the second column of $\mathbf{B}$ yields $c_{12} = 0$
$3b_{11} + 4b_{21} = 0$, second row of $\mathbf{A}$ multiplied with the first column of $\mathbf{B}$ yields $c_{21} = 0$
$3b_{12} + 4B_{22} = 1$, second row of $\mathbf{A}$ multiplied with the second column of $\mathbf{B}$ yields $c_{22} = 1$

This system of equations has to be solved to get the inverse matrix **B** of **A**

1.2.2 A method of calculating the inverse matrix

The system of equations to determine the inverse matrix **B** of **A** can be solved simultaneous for the unknown quantities b_{ik} in the following way:

write the unit matrix next to the coefficient matrix on the right side and convert the so obtained double matrix in a way, that the identity matrix is left standing at the end. Then on the right side will stand the inverse matrix.

$$\left(\begin{array}{cc|cc} 5 & 6 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$$

allowed transformations

(1) multiply a row with an arbitrary number
(2) add a row to another row
(3) It is also allowed to multiply a row with any number, to add the result to another row, and to let the row itself unchanged.

transformation of the double matrix

(1) multiply the first row with $+3$, the second row with -5

$$\left(\begin{array}{cc|cc} 15 & 18 & 3 & 0 \\ -15 & -20 & 0 & -5 \end{array}\right)$$

Explanation:

The first row of the double matrix
$$\left(\begin{array}{cc|cc} 5 & 6 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$$
is a row vector, specified as follows

$$\left(\begin{array}{cccc} 5 & 6 & 1 & 0 \end{array}\right)$$

Multiplying a row vector with a number means multiplying any component with this number

$(+3) \cdot \left(\begin{array}{cccc} 5 & 6 & 1 & 0 \end{array}\right)$ yields

$$\left(\begin{array}{cccc} 15 & 18 & 3 & 0 \end{array}\right)$$

the second row of the double matrix

$$\left(\begin{array}{cccc} 3 & 4 & 0 & 1 \end{array}\right)$$

Multiplication with -5 yields

$\begin{pmatrix} -15 & -20 & 0 & -5 \end{pmatrix}$

(2) add the first row to the second row

$$\left(\begin{array}{cc|cc} 15 & 18 & 3 & 0 \\ 0 & -2 & 3 & -5 \end{array}\right)$$

Explanation the specified transformation:

First the double matrix

$$\left(\begin{array}{cc|cc} 15 & 18 & 3 & 0 \\ -15 & -20 & 0 & -5 \end{array}\right)$$

is considered, which is the result of the transformation under (1).

The rows of this matrix are row vectors. Row vectors are added by adding the corresponding components:

$$\begin{pmatrix} 15 & 18 & 3 & 0 \end{pmatrix} + \begin{pmatrix} -15 & -20 & 0 & -5 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 3 & -5 \end{pmatrix}$$

Why these transformations?

One wishes to achieve, that the identity matrix stands on the left side of the double matrix, at the end of all transformations.
The so far achieved zero is already in place.

(3) multiply the second row with +9 and add the result to the first row

$$\left(\begin{array}{cc|cc} 15 & 0 & 30 & -45 \\ 0 & -2 & 3 & -5 \end{array}\right)$$

(4) divide the first row by 15 and the second row by −2

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & \frac{-3}{2} & \frac{5}{2} \end{array}\right)$$

in this way one gets the inverse matrix to

$$\begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$$

it is described as

$$\begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix}$$

2 Gaussian algorithm

2.1 The Gaussian algorithm for a (3,3) matrix

As an example, consider the following system of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3
\end{aligned}
$$

The coefficient matrix of this system is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Try to transform the coefficient matrix in an upper triangular form, i.e. below the diagonal of the matrix all components are zero.

The upper triangular form of a (3,3) matrix looks like

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix}$$

The transformations take place in the expanded matrix.

The expanded matrix for the above system of equations has the following form

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right)$$

The expanded matrix consists of the coefficient matrix and the result vector of the linear equation system.

Permissible transformations are

(1) Multiplying a row with an arbitrary number
(2) Adding a row to another row
(3) It is also allowed to multiply a row with any number, to add the result to another row and let the line itself unchanged.

The following form should arise as the result of the transformations:

$$\left(\begin{array}{ccc|c} c_{11} & c_{12} & c_{13} & d_1 \\ 0 & c_{22} & c_{23} & d_2 \\ 0 & 0 & c_{33} & d_3 \end{array}\right)$$

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