Sequences of numbers

1 Examples for sequences of numbers

1.1 harmonic sequence

Example: $a_n = \frac{1}{n}; n \geq 1; \{a_n\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} ...$

$\lim_{n\to\infty} a_n = 0$

1.2 alternating sequence

Example: $a_n = (-1)^n; n \geq 0; \{a_n\} = 1, -1, 1, -1...$

$\lim_{n\to\infty} a_n$ does not exist.

1.3 recursive presentation of a sequence

Example: $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right); n \geq 1; a_1 = 1$

$a_2 = \frac{3}{2}$

$a_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right); \; a_3 = \frac{17}{12} = 1,41667$

$a_4 = 1,41422$

Under the assumption, that the sequence a_n converges against a limit a^*, one can give an estimate of a^*:

$a^* = \frac{1}{2}\left(a^* + \frac{2}{a^*}\right)$

From the last equation it follows $2a^* = \frac{(a^*)^2 + 2}{a^*}$

$2(a^*)^2 = (a^*)^2 + 2, \; (a*)^2 = 2; a* = \sqrt{2}$

The proof of convergence to $\sqrt{2}$ is presented by the link below. This proof is not subject of the tutorial.
convergence against the square root of 2

1.4 arithmetic sequence

For an arithmetic sequence the following definition holds: $a_{n+1} - a_n = d; n \geq 0$. d is a constant real number.

Example: $a_0 = 2; a_1 = 4; a_2 = 6; a_3 = 8...$, it follows $a_{n+1} - a_n = 2$

1.5 geometric sequence

$\frac{a_{n+1}}{a_n} = q; n \geq 0; a_0 \neq 0; q > 0$

Example

$a_0 = 2; a_1 = 4 \Rightarrow \frac{a_1}{a_0} = 2;$

$a_2 = 8, a_3 = 16..., \frac{a_2}{a_1} = 2, \frac{a_3}{a_2} = 2$

$a_n = 2^{n+1}; n \geq 0$

$\frac{a_{n+1}}{a_n} = \frac{2^{n+2}}{2^{n+1}} = 2$

$q = 2$

2 Limit theorems

In the following it is assumed that for all limits n approaches ∞.

2.1 LT 1: $a_n \to a$ and $b_n \to b$, $a_n + b_n \to a + b$, $a_n \cdot b_n \to a \cdot b$

The sequence $\{a_n\}$ shall converge against a real number a and the sequence $\{b_n\}$ shall converge against a real number b. Then the sequence $\{a_n + b_n\}$ converges against the limit $a + b$ and the sequence $\{a_n \cdot b_n\}$ converges against $a \cdot b$.

Examples: $3 + \frac{2}{n}; \frac{6}{n}$

$a_n = 3, b_n = \frac{2}{n}$

$a_n + b_n = 3 + \frac{2}{n};\ a_n \cdot b_n = \frac{6}{n}$

$a = \lim_{n\to\infty} a_n;\ a = 3;\ b = \lim_{n\to\infty} b_n;\ b = 0$

$a + b = \lim_{n\to\infty} (a_n + b_n);\ a + b = 3$

$a \cdot b = \lim_{n\to\infty} (a_n \cdot b_n);\ a \cdot b = 0$

2.2 LT 2: $a_n \to a$ and $b_n \to b$, $b \neq 0$, $\frac{a_n}{b_n} \to \frac{a}{b}$

If the sequence $\{a_n\}$ converges against a real number a and the sequence $\{b_n\}$ converges against a real number $b \neq 0$, then the sequence $\{\frac{a_n}{b_n}\}$ converges against the real number $\frac{a}{b}$

Example: $\dfrac{3+\dfrac{2}{n}}{1+\dfrac{3}{n}}$

$a_n = 3 + \dfrac{2}{n}, b_n = 1 + \dfrac{3}{n}$

$3 + \dfrac{2}{n}$ converges against 3; $a = 3$

$1 + \dfrac{3}{n}$ converges against 1; $b = 1$

It follows

$\dfrac{3+\dfrac{2}{n}}{1+\dfrac{3}{n}}$ converges against $\dfrac{a}{b} = 3$

2.3 LT 3: $a_n \to \infty$ and $b_n \to b$, $a_n + b_n \to \infty$

Assumption: $\lim\limits_{n\to\infty} a_n = \infty$

If $\{b_n\}$ converges against a real number b, then it follows $\lim\limits_{n\to\infty}(a_n + b_n) = \infty$

Example:

$a_n = 4n, b_n = \dfrac{8}{n}$

$\lim\limits_{n\to\infty} 4n = \infty,\ \lim\limits_{n\to\infty} \dfrac{8}{n} = 0;\ b = 0$

it follows $\lim\limits_{n\to\infty}(a_n + b_n) = \infty$

2.4 LT 4: $a_n \to \infty$ and b_n gegen $b \neq 0$, $a_n \cdot b_n$

The product sequence $\{a_n \cdot b_n\}$ converges against $+\infty$, if $b > 0$ and against $-\infty$, if $b < 0$.

Example: $(1+n)\left(-3 + \dfrac{1}{n}\right)$

$a_n = 1 + n, b_n = \left(-3 + \dfrac{1}{n}\right)$

$\lim\limits_{n\to\infty} a_n = \infty,\ \lim\limits_{n\to\infty} b_n = -3;\ b = -3$

$\lim\limits_{n\to\infty}(a_n \cdot b_n) = -\infty$

additional calculation:

$(1+n)(-3+\dfrac{1}{n}) = -3 - 3n + \dfrac{1}{n} + 1$ converges against $-\infty$

2.5 LT 5: $a_n \to a$, $a \neq 0$, $b_n \to \infty$, $\frac{a_n}{b_n} \to 0$

Assumption: $\lim\limits_{n\to\infty} b_n = \infty$ and $\lim\limits_{n\to\infty} a_n = a$ with $a \neq 0$

Then it follows $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$

Example: $\frac{2+\frac{1}{n}}{3+n}$

$a_n = 2 + \frac{1}{n}$; $b_n = 3 + n$

$a_n \to 2$; $b_n \to \infty$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$

2.6 LT 6: $a_n \to \infty$ and $b_n \to$b; $b \neq 0$, $\frac{a_n}{b_n} \to \infty$

Assumption: $\lim\limits_{n\to\infty} a_n = \infty$ and $\{b_n\}$ converges against a real number $b \neq 0$.

Then it follows $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$

Example: $\frac{2+n^2}{1+\frac{1}{n}}$

$a_n = 2 + n^2$

$b_n = 1 + \frac{1}{n}$

$a_n \to \infty$; $b_n \to 1$; $b = 1$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$

2.7 LT 7: $a_n \to a$, $a \neq 0$ and $b_n \to 0$, $\frac{a_n}{b_n}$

Assumption: $\lim\limits_{n\to\infty} a_n = a$ and $a \neq 0$. The sequence $\{b_n\}$ converges against 0.

If $a > 0$ then $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$

If $a < 0$ then $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = -\infty$

Beispiel: $\frac{1+\frac{1}{n}}{\frac{1}{n}}$

$a_n = 1 + \frac{1}{n}, b_n = \frac{1}{n}$

$\lim_{n \to \infty} a_n = 1, \lim_{n \to \infty} b_n = 0$

It follows $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$

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